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An exact invariant for the time dependent double well anharmonic oscillators: Lie theory and quasi-invariance groups

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Abstract. We derive an exact energy type invariant for the double well anharmonic oscillators. This invariant holds for certain time dependent potential parameters. We use a nonlinear superposition principle to solve the nonlinear constraint equations, which appear in the resolution.

1. Introduction

Generally, differential equations with time dependent coefficients cannot be solved directly. It may be fertile to find invariants (conserved quantities) of the problem.

Some years ago, Lewis (1968) found an invariant of the time dependent harmonic oscillator. This invariant is expressed in terms of a particular solution of some auxiliary equation. Lewis and Reisenfeld (1969) used this invariant to give an exact quantum mechanical treatment of the time dependent harmonic oscillator. Later Ray and Reid (1979) and Lutzky (1980) extended this result to the so-called Ermakov system (Ermakov 1880). Morever this invariant is used to work out a nonlinear superposition law (Ray 1980), i.e. a relation which combine solutions in a certain way to generate new solutions, for the Ermakov systems.

On other hand, the Feix group has revived the interest of time dependent coordinate transformations for differential equations with time dependent coefficients. They have extended this kind of transformation, regarding it as an element of a quasi-invariance group, to treat non-Hamiltonian problems. They used such transformations to study asymptotic solutions in time-dependent problems such as plasma physics (Burgan *et al* 1978) Newtonian equation of motion (Moraux *et al* 1981) and quantum mechanics (Munier *et al* 1981).

In a recent paper Leach (1981) builds an exact invariant for a class of time dependent anharmonic oscillators with cubic anharmonicity perturbation. He uses the methods of the Lie theory of extended groups which is very similar to the quasi-invariance group method. Moreover he obtains the constraints on the time dependent coefficient of the perturbation, in order that an exact invariant should exist. Indeed, in general, a time dependent equation does not possess an invariant.

In this paper we study the time dependent quartic double well anharmonic oscillators. We give, for possible quantum applications, an invariant and constraints on the time dependent potential parameters. In § 2 we recall some basic features about the method of Lie group. In § 3 we give the expression of the energy-type invariant. In § 4 we show that this method is in fact a special case of the quasi-invariance group. Finally in § 5, we use a nonlinear superposition law to solve formally the equations giving the potential parameters.

2. Lie theory and second-order differential equations

We consider the problem of a classical particle in a time dependent quartic potential

$$V(q, t) = \tfrac{1}{2}\mu(t)q^2 + \tfrac{1}{4}\lambda(t)q^4, \tag{2.1}$$

where $\mu(t)$ and $\lambda(t)$ are two functions of time (t). If $\mu(t)$ is negative and $\lambda(t)$ positive, $V(q, t)$ has two minima located at $q = \pm(-\mu/\lambda)^{1/2}$ where $V(q, t)$ has the value $-\mu^2/4\lambda$. If $\mu(t)$ and $\lambda(t)$ have the same sign there is one extremum ($q = 0$, $V = 0$). Now, let us study, the equation of the motion of a particle, with a linear friction in the potential defined by (2.1)

$$\mathrm{d}^2q/\mathrm{d}t^2 + \delta(t)\,\mathrm{d}q/\mathrm{d}t + \mu(t)q + \lambda(t)q^3 = 0. \tag{2.2}$$

This equation can easily be converted into a form without linear friction. To prove this assertion we perform the following change to a new independent variable θ defined as

$$\mathrm{d}\theta = \mathrm{d}t/A^2(t), \tag{2.3}$$

where $A(t)$ is a function of t. The equation of motion (2.2) now reads

$$\frac{1}{A^4(t)}\frac{\mathrm{d}^2q}{\mathrm{d}\theta^2} + \frac{1}{A^2(t)}\left(\delta(t) - 2\frac{\dot{A}(t)}{A(t)}\right)\frac{\mathrm{d}q}{\mathrm{d}\theta} + \mu(t)q + \lambda(t)q^3 = 0. \tag{2.4}$$

In order to cancel the linear friction $\delta(t)$, $A(t)$ must verify the following equation

$$A(t) = C_{\mathrm{int}} \exp \tfrac{1}{2} \int \delta(t')\,\mathrm{d}t', \tag{2.5}$$

where C_{int} is an integration constant.

Thus, the case with linear friction can be assimilated to the case without friction. For this reason we shall consider the friction-free case for simplicity and with no loss of generality, although the Lie theory of extended group holds for an equation of motion with friction. In this section we use the Lie Theory of extended groups and derive the expression of the generator. Let

$$E(\ddot{q}, \dot{q}, q, t) = 0, \tag{2.6}$$

be a Newtonian equation of motion. Suppose equation (2.6) remains invariant under a transformation of generator U, we require

$$U^{(2)}E(\ddot{q}, \dot{q}, q, t) = 0, \tag{2.7}$$

where $U^{(2)}$ is the second extension of U given by

$$U^{(2)} = U + \eta^{(1)}\,\partial/\partial\dot{q} + \eta^{(2)}\,\partial/\partial\ddot{q}, \tag{2.8}$$

where

$$U=\xi(q,t)\,\partial/\partial t+\eta(q,t)\,\partial/\partial q, \qquad (2.8a)$$

$$\eta^{(1)}=\dot{\eta}-\dot{q}\dot{\xi}, \qquad \eta^{(2)}=\ddot{\eta}-\ddot{\xi}\dot{q}-2\dot{\xi}\ddot{q}.$$

Equation (2.7) gives, if $E\equiv\ddot{q}+g(q,t)=0$, a polynomial equation in $\dot{q}$. By equating coefficients of powers of $\dot{q}$, one obtains

$$\partial^2\xi/\partial q^2=0, \qquad (2.9a)$$

$$\partial^2\eta/\partial q^2-2\,\partial^2\xi/\partial q\,\partial t=0, \qquad (2.9b)$$

$$2\,\partial^2\eta/\partial t\,\partial q-\partial^2\xi/\partial t^2+3g\,\partial\xi/\partial q=0, \qquad (2.9c)$$

$$\xi\,\partial g/\partial t+\eta\,\partial g/\partial q+\partial^2\eta/\partial t^2-g\,\partial\eta/\partial q+2g\,\partial\xi/\partial t=0. \qquad (2.9d)$$

Integrating (2.9a) and (2.9b) gives:

$$\xi(q,t)=b(t)q+a(t), \qquad \eta(q,t)=\dot{b}(t)q^2+c(t)q+d(t). \qquad (2.10a)$$

For the special case, when (2.2) holds, from (2.9c) and (2.9d) one gets

$$\begin{aligned}&b=0, \qquad 2\dot{c}-\ddot{a}=0, \qquad a\dot{\mu}+2\dot{a}\mu+\ddot{c}=0,\\ &\dot{\lambda}a+2\lambda c+2\dot{a}\lambda=0, \qquad \ddot{d}+\mu d=0.\end{aligned} \qquad (2.10)$$

For simplicity we restrict ourselves to the case $d=0$. Then the solution of equations (2.10) is

$$\begin{aligned}&2c(t)=\dot{a}(t)+\alpha,\\ &\lambda(t)=Ka^{-3}(t)\exp\left(-\alpha\int^{t}\frac{dt'}{a(t')}\right),\end{aligned} \qquad (2.11)$$

where K and α are integration constants. Functions $a(t)$ and $\mu(t)$ are related by the third-order equation

$$\dddot{a}+2a\dot{\mu}+4\dot{a}\mu=0. \qquad (2.12)$$

Equations (2.11) and (2.12) clearly show that $\lambda(t)$ and $\mu(t)$ are not independent. Only one of them can be arbitrarily chosen, the other one is given by the solution of equations (2.11)–(2.12). Then, the second extension of U is now given by

$$U^{(2)}(\dot{q},q,t)=a\frac{\partial}{\partial t}+\tfrac{1}{2}q(\dot{a}+\alpha)\frac{\partial}{\partial q}+\tfrac{1}{2}[\ddot{a}q+(\alpha-\dot{a})\dot{q}]\frac{\partial}{\partial\dot{q}}+\tfrac{1}{2}[\dddot{a}q+(\alpha-3\dot{a})\ddot{q}]\frac{\partial}{\partial\ddot{q}}. \qquad (2.13)$$

3. Invariant of the problem

In order to find an invariant to the problem defined by equation (2.2), one way is to apply Noether's theorem which gives an invariant to any symmetry type in the Lagrangian. In our case, it is given by

$$L(\dot{q},q,t)=\tfrac{1}{2}\dot{q}^2+V(q,t). \qquad (3.1)$$

Noether's theorem states that if

$$U^{(1)}(\dot{q},q,t)L(\dot{q},q,t)=0, \qquad (3.2)$$

where $U^{(1)}$ is the first extension of the vector field U, then the quantity

$$\Phi(\dot{q}, q, t) = (\xi\dot{q} - \eta)\,\partial L/\partial\dot{q} - \xi L + f, \tag{3.3}$$

is conserved (Gelfand and Fomin 1963), where $f(q, t)$ is a function determined together with ξ and η.

Another way to find an energy type invariant is to use the fact that the invariance of energy is associated to the time translational invariance (Leach 1981), then, the generator of such a transformation is

$$U = \partial/\partial t. \tag{3.4}$$

Consequently, with a generalised canonical transformation

$$T = f(t), \qquad Q = g(t)q + h(t), \tag{3.5}$$

which is linear in q and nonlinear in t, we can determine the functions f, g, and h under the condition

$$\tilde{U}(Q, T) = \partial/\partial T, \tag{3.6}$$

where $\tilde{U}$ is U expressed in terms of the new variables (Q, T). Inserting the transformation (3.5) in the expression (2.8A) for $U(q, t)$ and assuming the condition (3.6) is verified, one finds

$$a\dot{h} = 0, \qquad a\dot{f} = 1, \qquad a\dot{g} + \tfrac{1}{2}(\dot{a} + \alpha)g = 0, \tag{3.7}$$

and by integration, we obtain

$$h(t) = C_1, \qquad f(t) = C_2 \int_0^t \frac{\mathrm{d}t'}{a(t')}, \qquad g(t) = C_3 a^{-1/2}(t) \exp\left(-\frac{\alpha}{2}\int_0^t \frac{\mathrm{d}t'}{a(t')}\right). \tag{3.8}$$

To determine the integration constants C_1, C_2 and C_3 it is required that the coordinates (q, t) and (Q, T) coincide at $t = 0$, if $a(0) = 1$, then $C_1 = 0$ and $C_2 = C_3 = 1$. We can now write the equation (2.1) in terms of coordinate Q and time T,

$$\mathrm{d}^2Q/\mathrm{d}T^2 + \alpha\,\mathrm{d}Q/\mathrm{d}T + MQ + NQ^3 = 0, \tag{3.9}$$

where

$$\begin{aligned} M &= \tfrac{1}{2}(\dot{a} + \alpha)^2 - \tfrac{1}{4}(-2a\ddot{a} + 4\alpha\dot{a} + 3\dot{a}^2 + \alpha^2) + \mu a^2, \\ N &= \lambda(t)a^3 \exp(\alpha f). \end{aligned} \tag{3.10}$$

The last equation, for N, is nothing but equation (2.11) where K appears as an integration constant. Evidently, N is a constant. Using the contraint equation (2.12) it is straightforward to check that M is also a constant.

We can now notice that the time dependent coordinate transformation (3.5) with equations (3.8) give the following transformation law of phase space element

$$\mathrm{d}q\,\mathrm{d}v = \exp\left(-\frac{\alpha}{2}\int^t \frac{\mathrm{d}t'}{a(t')}\right)\mathrm{d}Q\,\mathrm{d}V, \tag{3.11}$$

where $v = \mathrm{d}q/\mathrm{d}t$ and $V = \mathrm{d}Q/\mathrm{d}T$. This equality (3.11) clearly shows that the Hamiltonian formalism occurs only for $\alpha = 0$. Under these circumstances equation (3.9) can be integrated and gives the energy type invariant:

$$\Phi(Q', Q) = \tfrac{1}{2}Q'^2 + \tfrac{1}{2}MQ^2 + \tfrac{1}{4}NQ^4, \tag{3.12}$$

where the prime denotes differentation with respect to the new time T. This dual potential is now time independent. It has two minima if M is negative and N is positive. This last condition requires a $(t)>0$ in agreement with equations (3.8). Equations (3.10) are now differential equations

$$\tfrac{1}{2}a\ddot{a}-\tfrac{1}{4}\dot{a}^2+\mu a^2=M, \qquad \lambda(t)=Na^{-3}, \tag{3.13}$$

where M and N are two constants.

4. Quasi-invariance group

In this section we apply the quasi-invariance group method to the time dependent double-well oscillator. We introduce a rescaling of space and time as follows

$$\mathrm{d}\theta=\mathrm{d}t/A^2(t), \qquad q(t)=X(\theta)C(t), \tag{4.1}$$

where X and θ are respectively the new coordinate and time. A and C are two functions of time t.

Applying this transformation to equation (2.2) we obtain

$$\frac{\mathrm{d}^2X}{\mathrm{d}\theta^2}+A^2\left[2\left(\frac{\dot{C}}{C}-\frac{\dot{A}}{A}\right)+\delta\right]\frac{\mathrm{d}X}{\mathrm{d}\theta}+X\frac{A^4}{C}(\ddot{C}+\delta\dot{C}+\mu C)+\lambda X^3A^4C^2=0. \tag{4.2}$$

This equation is very similar to (3.9). However we note that the transformation renormalises the friction term by a time dependent coefficient. The friction term $\delta(t)$ can be eliminated straightforwardly by a choice on A and C as done in § 2. This leads to a differential equation for C/A and the solution is

$$C(t)=A(t)\exp\left(-\tfrac{1}{2}\int_0^t\delta(t')\,\mathrm{d}t'\right), \tag{4.3}$$

where the integration constant has been chosen to verify $A(0)=C(0)$. Under the conditions

$$\ddot{C}+\delta(t)\dot{C}+\mu(t)C=\Omega C/A^4, \qquad \lambda(t)A^4C^2=k, \tag{4.4}$$

where k and Ω are constants, equation (4.2) now leads to the Hamiltonian of the particle in a potential

$$V(X)=\tfrac{1}{2}\Omega X^2+\tfrac{1}{4}kX^4. \tag{4.5}$$

This problem has an invariant which is obtained by integration with respect to X

$$I(X,X')=\tfrac{1}{2}(\mathrm{d}X/\mathrm{d}\theta)^2+\tfrac{1}{2}\Omega X^2+\tfrac{1}{4}kX^4. \tag{4.6}$$

It can be emphasised that this invariant does not exist for an arbitrary triplet ($\mu(t)$, $\lambda(t)$, $\delta(t)$). Only two of these functions can be chosen arbitrarily, the third being submitted to the constraints (4.4). Also the change of function $c(t)=\sqrt{a(t)}$ in system (4.4) with $\delta=0$ leads to

$$a\ddot{a}-\tfrac{1}{2}\dot{a}^2+2\mu a^2=2\Omega, \tag{4.7}$$

which is nothing but the equations relating the potential parameters obtained in § 3 with $M=\Omega$.

We now deal with the asymptotic solutions. When the friction is positive, (when $\delta=0$, this occurs if $C(t)$ increases more rapidly that $A(t)$) the asymptotic solution is one the two minima of the potential (4.5)

$$X_{as} = \pm(-\Omega/k)^{1/2} \qquad \text{or in } q\text{-space } q_{as} = X_{as}C(t). \tag{4.8}$$

To conclude this section we make a remark about the change of variables (4.1). This linear change of variable in X is the only one which gives an Hamiltonian formalism. Indeed, let us generalise the time dependent change of variables following

$$q(t) = X(\theta)C(t) + X^2(\theta)B(t). \tag{4.9}$$

The phase-space is transformed as

$$\mathrm{d}q\,\mathrm{d}v = A^{-2}(C + 2XB)^2\,\mathrm{d}X\,\mathrm{d}V, \tag{4.10}$$

which is invariant only if $A = C$ and $B = 0$.

However we cannot use this change of variable study the non-Hamiltonian problem (2.2). The equation of motion in dual space contains a friction term of the form $(B/C)(\mathrm{d}X)^2/(\mathrm{d}\theta)^2$ which is always positive, under the reflection $X \to -X$. This is not possible for a friction term and we must require $B = 0$. Then equation (4.9) reduces to the linear change of variables.

5. Solution of the potential parameters constraint

In order to give the explicit form of the invariant we must solve the system (4.4) where $\mu(t)$ and $\lambda(t)$ are given functions of time. If $\lambda(t)$ is known it is straightforward to obtain $C(t)$ and $\mu(t)$. On the contrary if $\mu(t)$ is the given function, we must solve, in a first step the nonlinear equation (4.4). Acrording to the remark, of § 1, about the friction term, we solve (4.4) in the special case $\delta = 0$. To solve it we use a so-called nonlinear superposition law. Let us consider the following system

$$\begin{aligned} &\ddot{C} + \mu(t)C = \Omega/C^3, \\ &\ddot{X} + \mu(t)X = 0, \end{aligned} \tag{5.1}$$

which is called an Ermakov system (Ermakov 1880).

The first equation is equation (4.4) and second one an auxiliary equation. For such a system, the nonlinear superposition law has been given by Ray (1980). Let us recall the derivation of this principle.

In a first step one constructs an invariant of (5.1) by eliminating, for example, the function $\mu(t)$ and integration with respect to t. One gets

$$J = \tfrac{1}{2}(C\dot{X} - X\dot{C})^2 + V(r), \tag{5.2}$$

where $V(r) = \Omega/2r^2$, with $r = C/X$.

Then we define a third time by $\mathrm{d}\tau = \mathrm{d}t/X^2(t)$. In terms of these variables, the invariant takes the form

$$J = \tfrac{1}{2}(\mathrm{d}r/\mathrm{d}\tau)^2 + \Omega/2r^2. \tag{5.4}$$

Integration leads to

$$\tau + \tau_0 = \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}r}{\sqrt{J - V(r)}}, \tag{5.5}$$

where τ_0 is an arbitrary constant. This equation leads to the nonlinear superposition law. If we known any particular solution $X(t)$ of equation (5.1) the general solution $C(t)$ is

$$C(t)=X(t)r\left(\int \mathrm{d}t/X^2(t)+\tau_0, J\right), \tag{5.6}$$

where τ_0 and J are two integration constants.

In our case, we obtain r from equation (5.5). Then (5.6) is in fact

$$C(t)=X(t)\sqrt{\frac{\Omega}{2J}}\left[\frac{4J^2}{\Omega}\left(\int \mathrm{d}t/X^2(t)+\tau_0\right)^2+1\right]^{1/2}, \tag{5.7}$$

where $C(t)$ is the general solution of (5.1) and $X(t)$ a particular solution.

6. Concluding remarks

We have considered the time dependent anharmonic double well oscillators. We have constructed an energy type invariant of this problem in the special case where the potential are solutions of a nonlinear differential system. We have given the formal solution of this equation, using a nonlinear superposition law.

We have seen that the formalism of quasi-invariance group is more general and more tractable than the Lie theory of extended group. Moreover the formalism is well suited to the non-Hamiltonian case and to the asymptotic study, by the choice left on the friction term.

Finally we have shown that the generalised canonical transformation is the most general that we can use for a physical problem.

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